ARE WE COUNTING OR MEASURING SOMETHING?

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ABSTRACT. Let H be a semisimple Hopf algebras over an algebraically closed field k of characteristic 0. We define Hopf algebraic analogues of commutators and their generalizations and show how they are related to H', the Hopf algebraic analogue of the commutator subgroup. We introduce a family of central elements of H', which on one hand generate H' and on the other hand give rise to a family of functionals on H. When $H = kG$, G a finite group, these functionals are counting functions on G. It is not clear yet to what extent they measure any specific invariant of the Hopf algebra. However, when H is quasitriangular they are at least characters on H.

INTRODUCTION

Commutators and commutator subgroups are some of the most fundamental concepts in group theory. These subgroups measure how far the group is from being commutative. It was Frobenius who proved early on that a function on a finite group G, that counts the number of ways an element of G can be realized as a commutator, is a character of G. This was done by giving an explicit formula for this counting function. Generalizations of this formula were proved throughout the years (e.g. [Ta, KM, AV]). The approaches varied from a direct approach through the use of a symmetric bilinear form and its associated Casimir element, to the use of distribution functions which are uniform on conjugacy classes of G.

In this paper we define Hopf algebraic analogues of commutators and their generalizations and show how they are related to H', the Hopf algebraic analogue of the commutator subgroup. We introduce a family of elements in H' denoted by z_n, $n > 1$, which arise from the idempotent integral of H. This family consists of powers of the S-fixed central invertible element z_2,

$$z_2 = \sum \Lambda_1 \Lambda_1' S \Lambda_2 S \Lambda_2'$$

where Λ, Λ' are two copies of the idempotent integral of H.

The elements $\{z_n\}$ are shown to give rise to Hopf algebraic analogues of the various counting functions for groups. On the other hand they are shown to be central Casimir elements associated to certain symmetric bilinear forms and Higman maps on H or on its center.

A different characterizations of z_n is given when H is also assumed to be almost cocommutative. In this situation the z_n's are related to so called "generalized class sums" for H. Information about functionals related to iterated commutators can be deduced from the character table of H.

We use the following notations. Let H be a d-dimensional semisimple Hopf algebras over an algebraically closed field k of characteristic 0 with an idempotent integral Λ. Denote by $\{E_i\}_{0 \leq i \leq n-1}$ the set of primitive central idempotents of H. Let d_i denote the degree of the irreducible character χ_i and let $R(H) = \mathrm{Sp}_k\{\chi_i\}$. Then $\chi_H = \lambda = \sum d_i \chi_i$ is an integral for H^*.

Let $\Psi : H_{H^*} \to H^*_{H^*}$ be the Frobenius map given by:

$$\Psi(h) = \lambda \leftharpoonup S(h).$$

The commutator subalgebra H' is a normal left coideal subalgebra of H for which $H/(HH'^+)$ is commutative and it is minimal with respect to this property.

In Section 2 we define the commutator $\{a, b\}$ for $a, b \in H$ as follows:

$$\{a, b\} = \sum a_1 b_1 S a_2 S b_2.$$

We define also the general commutator $\{a^1 \ldots, a^n\}$, $a^i \in H$ as follows:

$$\{a^1, \ldots, a^n\} = \sum a_1^1 \cdots a_1^n S a_2^1 \cdots S a_2^n.$$

We show that the general commutator can always be obtained as a sum of products of commutators.

Of special interest are commutators related to the idempotent integral Λ of H. Let Λ^i be a copy of the idempotent Λ. Set

$$z_n = \sum \Lambda_1^1 \cdots \Lambda_1^n S \Lambda_2^1 \cdots S \Lambda_2^n \qquad z_0 = 1.$$

We show:

Theorem 2.7:Let H be a semisimple Hopf algebra over an algebraically closed field k of characteristic 0. Then for all $k, n \geq 0$,

$$z_{2k+1} = z_{2k} \qquad z_n = z_2^{\frac{n - n(\mathrm{mod}2)}{2}} \in Z(H).$$

The commutator z_2 has a very nice form:

Theorem 2.8: Let H be a semisimple Hopf algebra over an algebraically closed field k of characteristic 0. Then

$$z_2 = \sum \Lambda_1^1 \Lambda_1^2 S\Lambda_2^1 S\Lambda_2^2 = \sum_i \frac{1}{d_i^2} E_i \in Z(H).$$

The elements z_n serve as indicators for the commutativity of H as follows:

Theorem 2.10: Let H be a semisimple Hopf algebra over an algebraically closed field k of characteristic 0. Then H is commutative if and only if $z_n \in k$ for some $n \geq 2$, if and only if $z_n \in k$ for all $n \geq 2$.

In fact z_n generates H' in the following sense:

Theorem 2.12: Let H be a semisimple Hopf algebra over an algebraically closed field k of characteristic 0 and let $n \geq 2$. Then the commutator sublagebra of H is the algebra generated by the left coideal

$$H_n' = z_n \leftharpoondown H^*.$$

In Section 3 we discuss "counting functions" for H. An important set of functionals on finite groups are the so called counting functions. As in [AV], let w be a word in the free groupg on r letters. Substituting $(g_1, \cdots, g_r) \to w(g_1, \ldots, g_r)$ defines a natural function from G^r to G. Let $N_w(g)$ denote the number of solutions to $w(g_1, \ldots, g_r) = g$. The function N_w is a class function, called a counting function.

We discuss analogues of the four following counting functions: (i)Root functions. (ii) Frobenius counting function for commutators. (iii) Counting functions for generalized commutators. (iv) Counting functions for iterated commutators.

The Frobenius counting function for commutators,

$$f_{rob} = \sum_i \frac{d}{d_i} \chi$$

is realized in our setup as:

$$f_{rob} = d\Psi(z_2).$$

More generally, the convolution power of f_{rob} is the counting function for the generalized commutators, and for products of commutators. The convolution product $\bullet$ inside H^* is defined for $p, q \in H^*$ as follows:

$$p \bullet p' = d\Psi(\Psi^{-1}(p)\Psi^{-1}(q)).$$

We show:

Theorem 3.3: Let H be a semisimple Hopf algebra over an algebraically closed field k of characteristic 0. Then for any $l > 0$,

$$d^{2l-1}\Psi(z_{2l}) = \sum_i \frac{d^{2l-1}}{d_i^{2l-1}}\chi_i = \underbrace{f_{rob} \bullet f_{rob} \cdots \bullet f_{rob}}_{l}.$$

The function in the middle is called f_n and for groups it equals the function that counts general commutators of length $2n$ or $2n + 1$. It also equals the function that counts products of n commutators.

The commutators z_n are realized also as Casimir elements for specific symmetric forms on H and on $Z(H)$.

The analogue of the counting function for iterated commutators of a group is defined as:

$$f_{\{\{x,y\},z\}} = d^2 \sum_{i,j} \frac{1}{d_i d_j}\langle \chi_i s(\chi_i)\chi_j, \Lambda\rangle\chi_i.$$

We show:

Theorem 3.7: Let H be a semisimple hopf algebra over an algebraically closed fiels of characteristic 0. Then:

$$f_{\{\{x,y\},z\}} = d^2 \sum_{i,j} \frac{1}{d_i}\langle \chi_i s(\chi_i), z_2\rangle\chi_i = d^2\Psi(\{z_2, \Lambda\}).$$

In Section 4 we focus on almost cocommutative Hopf algebras. Let $\{\eta_j\}$ be the set of normalized class sums of H.

Observe that for a group G, we have g is a commutator if and only if $f_{rob}(g) \neq 0$. Moreover, $g \in G'$ if and only if $f_n(g) \neq 0$ for some $n \geq 1$. Similarly, g is an iterated commutator if and only if $f_{\{\{x,y\}z\}}(g) \neq 0$. Hence also for a class sum C, $f_{rob}(C) \neq 0$ (respectively $f_n(C) \neq 0$, $f_{\{\{x,y\}z\}}(C) \neq 0$) if and only if C is a sum of commutators (respectively $C \in kG'$, C is a sum of iterated commutators).

In this spirit we show:

Theorem 4.4: Let H be an almost cocommutative semisimple Hopf algebra over an algebraically closed field of characteristic 0, let η_j be a normalized class sum and let f_n be as in (20). Then

$$\eta_j \in z_2^n \leftharpoonup H^* \iff \langle f_n, \eta_j\rangle \neq 0.$$

For Hopf algebras that satisfy the condition in Kaplansky's conjecture, that is, if $d_i|d$ for all i, we have:

Theorem 4.7: Let H be a semisimple Hopf algebra over an algebraically closed field of characteristic 0, and assume $d_i | d$ for any irreducible H-module V_i. Then the functions f_{rob}, f_n and $f_{\{\{x,y\},z\}}$ are characters.

1. PRELIMINARIES

Throughout this paper H is a d-dimensional Hopf algebra over an algebraically closed field k of characteristic 0. We denote by S and s the antipodes of H and H^* respectively and Λ and λ the left and right integrals of H and H^* respectively so that $\langle \lambda, \Lambda \rangle = 1$. Denote by $Z(H)$ the center of H.

For any algebra A, $A^* = \hom(A, k)$ becomes a right and left A-module by the *hit* actions $\leftharpoonup$ and $\rightharpoonup$ defined for all $a \in A$, $p \in A^*$,

$$\langle p \leftharpoonup a, a' \rangle = \langle p, aa' \rangle \qquad \langle a \rightharpoonup p, a' \rangle = \langle p, a'a \rangle$$

In particular If $A = H$ is a finite dimensional Hopf algebras then H^* is an algebra and thus H becomes a left and right H^* module.

Denote by $_{\dot{ad}}$ the left adjoint action of H on itself, that is, for all $a, h \in H$,

$$h_{\dot{ad}} a = \sum h_1 a S(h_2).$$

Then

$$\Lambda_{\dot{ad}} H \subset Z(H)$$

and if H is semisimple then equality holds.

A subalgebra A of H is called **normal** if it is stable under the left adjoint action.

Let $D(H)$ denote the Drinfeld double of the Hopf algebra H. It is not hard to see that H is a $D(H)$-module with respect to the adjoint action of H on itself and the right hit action of H^* on H.

Denote by $R(H)$ the k-span of all irreducible characters of H. It is an algebra (called the character algebra) which is contained in the algebra of all cocommutative elements of H^* and equal to it when H is semisimple.

Recall that H is a Frobenius algebra. One defines a Frobenius map $\Psi : H_{H^*} \to H_{H^*}^*$ by

(1) $$\Psi(h) = \lambda \leftharpoonup S(h)$$

where H^* is a right H^*-module under multiplication and H is a right H^*-module under right *hit*. If H is semisimple then

$$\Psi(Z(H)) = R(H).$$

For a finite-dimensional Hopf algebra H we have for all $p \in H^*$,

$$\Psi^{-1}(p) = \Lambda \leftharpoonup p,$$

Let H be a semisimple Hopf algebra over k. Let $\{k = V_0, \ldots V_{n-1}\}$ be a complete set of non-isomorphic irreducible H-modules, $\mathrm{Irr}(H) = \{\varepsilon = \chi_0, \ldots \chi_{n-1}\}$ the corresponding characters and $\{E_0, \ldots E_{n-1}\}$ the associated central primitive idempotents of H where $E_0 = \Lambda$ is the idempotent integral of H. Let $\dim V_i = d_i$, then

$$\lambda = \chi_H = \sum_{i=0}^{n-1} d_i \chi_i,$$

where χ_H denotes the character of the left regular representation.

A well known result of Larson [La] is the orthogonality of characters, that is,

$$(2) \qquad\qquad \langle \chi_i s(\chi_j), \Lambda \rangle = \delta_{ij}$$

The orthogonality of characters implies in particular (see also [Sc, Cor.4.6]):

$$(3) \qquad \langle \chi_i, E_j \rangle = \delta_{ij} d_j, \quad \Psi(E_i) = d_i s(\chi_i), \quad \chi_i \leftharpoonup E_j = \delta_{ij} \chi_i$$

for all $0 \leq i, j \leq n - 1$. In particular, $\{\chi_i\}$, $\{\frac{1}{d_j} E_j\}$ are dual bases of $R(H)$ and $Z(H)$ respectively. Hence we have for each $z \in Z(H)$, $p \in R(H)$

$$(4) \qquad\qquad z = \sum_i \frac{1}{d_i} \langle \chi_i, z \rangle E_i \qquad p = \sum_i \frac{1}{d_i} \langle p, E_i \rangle \chi_i.$$

By (3) we have

$$(5) \qquad\qquad \chi_i \leftharpoonup z = \frac{1}{d_i} \langle \chi_i, z \rangle \chi_i$$

for all i.

A finite dimensional algebra A over k is a **symmetric algebra** if there exists a non-degenerate associative symmetric bilinear form $\beta :$ $A \otimes A \to k$.

It is well known there exists a bijective correspondence between symmetric bilinear forms β on A and elements $t_\beta \in A^*$ so that $\langle t_\beta, ab \rangle = \langle t_\beta, ba \rangle$ for all $a, b \in A$ and $t_\beta \leftharpoonup A = A^*$, (that is t_β is a generator of A^* as an A-module). The correspondence is given as follows: Given t, a genrator of A^*, we define $\beta_t(a, b) = \langle t, ab \rangle$ and conversely, given β the the associated generator is $t_\beta \in A^*$ by $\langle t_\beta, a \rangle = \beta(a, 1)$.

For each β there exists $\{r_i\}$, $\{l_i\}$ which forms a dual basis for β, that is $\beta(r_i, l_j) = \delta_{ij}$, for $1 \leq i, j \leq \dim A$.

The **Casimir element** with respect to β is

$$\sum_i r_i \otimes l_i = \sum l_i \otimes r_i.$$

It does not depend on the choice of the dual basis for β. Moreover, for all $a \in A$,

$$(6) \qquad \sum_i r_i a \otimes l_i = \sum r_i \otimes a l_i \quad and \quad \sum a r_i \otimes l_i = \sum r_i \otimes l_i a$$

The **central Casimir** element with respect to β denoted by Cas_β is

$$Cas_\beta = \sum_i r_i l_i \in Z(A).$$

Any other symmetric form β' with corresponding t', r_i', l_i' satisfies $t' = t \leftharpoonup u$ where u is an invertible element in the center of A and

$$(7) \qquad r_i' = r_i u^{-1}, \ l_i' = l_i, \ Cas_{\beta'} = \sum r_i u^{-1} l_i.$$

Any symmetric form β defines an Higman map $\tau_\beta : H \to Z(H)$ by:

$$(8) \qquad \tau(h) = \sum r_i h l_i$$

Finite dimensional semisimple algebras are always symmetric. Also, if H be a semisimple Hopf algebra over an algebraically closed field k then it was shown in[OS] that H has a natural symmetric bilinear form β given by:

$$(9) \qquad \beta(h, h') = \langle \lambda, hh' \rangle,$$

for all $h, h' \in H$. The corresponding Casimir element is

$$(10) \qquad Cas_\beta = \sum \Lambda_1 \otimes S\Lambda_2.$$

It follows from (6) (it can also be implied directly from [LaRa, Lemma 1.2]) that for all $a \in H$,

$$(11) \qquad (i) \quad \sum \Lambda_1 \otimes aS\Lambda_2 = \sum \Lambda_1 a \otimes S\Lambda_2$$

$$(ii) \quad \sum a\Lambda_1 \otimes S\Lambda_2 = \sum \Lambda_1 \otimes S\Lambda_2 a.$$

2. Commutators for Hopf algebras

The commutator subalgebra H' of a semisimple Hopf algebra H was first defined in [Bu1]. It is a normal left coideal subalgebra of H for which $H/(HH'^+)$ is commutative and it is minimal with respect to this property. For $H = kG$ one has $H' = kG'$.

Based on [CW5, Prop.1.14], it is not hard to see that

$$(12) \qquad H' = \{h \in H \,|\, \sigma \rightharpoonup h = h \;\forall \sigma \in G(H^*)\}.$$

Generalizing from groups we describe H' in terms of Hopf algebraic commutators. Let H be any Hopf algebra over k. For $a, b \in H$, define their commutator $\{a, b\}$ as:

$$(13) \qquad \{a, b\} = \sum a_1 b_1 S a_2 S b_2.$$

It is straightforward to check that

$$(14) \qquad ab = \sum \{a_1, b_1\} b_2 a_2$$

for all $a, b \in H$. Define

$$(15) \qquad Com = \mathrm{span}_k\{\{a, b\} \,|\, a, b \in H\}.$$

Lemma 2.1. *Let H be a Hopf algebra over k, then Com is a left coideal of H.*

Proof. For each commutator $\{a, b\}$,

$$(16) \qquad \Delta(\{a, b\}) = \sum a_1 b_1 S a_3 S b_3 \otimes \{a_2, b_2\} \in H \otimes Com.$$

$\square$

As a result we show:

Proposition 2.2. *Let H be a semisimple Hopf algebra then the commutator subalgebra H' of H is the algebra generated by Com. Thus H is commutative if and only if $H' = k$.*

Proof. Let N denote the algebra generated by Com. Since Com is a left coideal, it follos that N is a left coideal subalgebra.

Next we show that N is normal. Since H is an H-module algebra under the left adjoint action it is enough to check it on Com. Indeed, by (14) we have for all $h, x, y \in H$,

$$\sum \underbrace{h_1\{x, y\}}_{a} \underbrace{Sh_2}_{b} = \sum \{h_1\{x, y\}_1, Sh_4\} Sh_3 h_2 \{x, y\}_2 = \sum \{h_1\{x, y\}_1, Sh_2\}\{x, y\}_2.$$

The first term is a commutator by definition, while the second one belongs to Com by Lemma 2.1.

Now, for any $\sigma \in G(H^*)$, we have:

$$\sigma \rightharpoonup \sum a_1 b_1 S a_2 S b_2 = \sum \langle \sigma, a_2 b_2 S a_3 S b_3 \rangle\, a_1 b_1 S a_4 S b_4 = \sum a_1 b_1 S a_2 S b_2$$

The last equality follows from the fact that σ is multiplicative on H. By the definition of H' we have that $Com \subset H'$ and since H' is an algebra, we have also $N \subset H'$.

Let $\overline{H} = H/HN^+$. Then by (14) we have for all $\overline{a}, \overline{b} \in \overline{H}$,

$$\overline{ab} = \sum \{\overline{a_1, b_1}\}\overline{b_2 a_2}.$$

But $\{\overline{a_1, b_1}\} = \langle \varepsilon, a_1 b_1 \rangle$, hence $\overline{ab} = \overline{ba}$. $\qquad\square$

Of special interest is the following commutator:

Lemma 2.3. *Let $z \in Z(H)$. Then $\{z, \Lambda\} \in Z(H)$.*

Proof. For all $h \in H$ we have:

$$h \sum z_1 \Lambda_1 S z_2 S \Lambda_2 =$$
$$= \sum h_1 z_1 \Lambda_1 S(Sh_3 h_2 z_2) S \Lambda_2$$
$$= \sum z_1 h_1 \Lambda_1 S(Sh_3 z_2 h_2) S \Lambda_2 \qquad (\text{since } z \in Z(H))$$
$$= \sum z_1 h_1 \Lambda_1 S h_2 S z_2 h_3 S \Lambda_2$$
$$= \sum z_1 h \Lambda_1 S z_2 S \Lambda_2 \qquad (\text{by (11)(i) with } a = h_3)$$
$$= \sum z_1 \Lambda_1 S z_2 S \Lambda_2 h \qquad (\text{since } \Lambda \cdot S z_2 \in Z(H))$$

$\qquad\square$

We consider now generalized commutators.

Definition 2.4. For Hopf algebras H and $n > 1$, the **n-th commutator** $\{a^1, \ldots, a^n\}$ where $a^1, \ldots a^n \in H$, is given by:

$$\{a^1, \ldots, a^n\} = \sum a_1^1 \cdots a_1^n S a_2^1 \cdots S a_2^n.$$

Define also the subspace generated by n-th commutators:

$$Com_n = \text{span}_k \{\{a^1, \ldots, a^n\} \mid a_i \in H \text{ for all } 1 \leq i \leq n\}.$$

Note Com_2 coincides with the subspace of commutators, Com. Moreover, $Com_n \subseteq Com_m$ for all $2 \leq n \leq m$. Hence proposition 2.2 can be generalized and we obtain:

Proposition 2.5. *Let H be a semisimple Hopf algebra. Then the commutator subalgebra H' of H is the algebra generated by Com_n. In particular, every generalized commutator is a linear combination of products of commutators.*

Proof. Since $Com_2 \subseteq Com_n$ it follows that the algebra generated by Com_2 is contained in the algebra generated by Com_n. By Proposition 2.2 it follows that H' is contained in the algebra generated by Com_n. Conversely, a direct computation shows that for any $\sigma \in H^*$,

$$\sigma \rightharpoonup \{a^1, \ldots, a^n\} = \{a^1, \ldots, a^n\}.$$

Hence equality holds.

The last part is straightforward.   □

Of special interest will be commutators and generalized commutators related to the idempotent integral Λ of H. Let Λ^i be a copy of the idempotent Λ. Set

$$(17) \qquad z_n = \sum \Lambda_1^1 \cdots \Lambda_1^n S\Lambda_2^1 \cdots S\Lambda_2^n \qquad z_0 = 1.$$

Define a map $Z_n : H \longrightarrow H$ by:

$$(18) \qquad Z_n(h) = \sum \Lambda_1^1 \cdots \Lambda_1^n h S\Lambda_2^1 \cdots S\Lambda_2^n \qquad Z_0(h) = h.$$

We show now:

Proposition 2.6. *Let H be a semisimple Hopf algebra. Then:*

1. *For all $h \in H$, $n > 1$*

$$Z_n(h) = z_2 Z_{n-2}(h).$$

2. *$Z_{2k+1}(h) \in Z(H)$ and $Z_{2k}(h) = z_{2k}h$ for all $k \geq 0$, $h \in H$.*

Proof. 1.

$Z_n(h) =$

$$= \sum \Lambda_1^1 \Lambda_1^2 \underbrace{\Lambda_1^3 \cdots \Lambda_1^n h}_{a} S\Lambda_2^1 S\Lambda_2^2 \cdots S\Lambda_2^n$$

$$= \sum \Lambda_1^1 \underbrace{\Lambda_1^3 \cdots \Lambda_1^n h}_{a} \Lambda_1^2 S\Lambda_2^1 S\Lambda_2^2 \cdots S\Lambda_2^{2n} \qquad \text{(by (11)(i) with } \Lambda = \Lambda^1)$$

$$= \sum \Lambda_1^1 \Lambda_1^2 S\Lambda_2^1 S\Lambda_2^2 \underbrace{\Lambda_1^3 \cdots \Lambda_1^n h}_{a} S\Lambda_2^3 \cdots S\Lambda_2^{2n} \qquad \text{(by (11)(ii) with } \Lambda = \Lambda^2)$$

$$= z_2 Z_{n-2}(h)$$

2. To see the odd case we first show that $z_2 \in Z(H)$. Indeed, by (11)(ii) with $\Lambda = \Lambda^1$ and then (11)(i) with $\Lambda = \Lambda^2$, we have for all $h \in H$,

$$hz_2 = \sum h\Lambda_1^1 \Lambda_1^2 S\Lambda_2^1 S\Lambda_2^2 = \sum \Lambda_1^1 \Lambda_1^2 S\Lambda_2^1 hS\Lambda_2^2 = \sum \Lambda_1^1 \Lambda_1^2 h S\Lambda_2^1 S\Lambda_2^2 = Z_2(h) = z_2 h.$$

The last equality follows from part 1.

Since $Z_1(h) = \Lambda_{ad}h \in Z(H)$, the result follows from part 1 by induction on k.

The even case follows by induction since $Z_0(h) = h$.   □

As a consequence we obtain the following theorem:

Theorem 2.7. *Let H be a semisimple Hopf algebra over an algebraically closed field k of characteristic 0. Then for all $k, n \geq 0$,*

$$z_{2k+1} = z_{2k} \qquad z_n = z_2^{\frac{n - n(\mathrm{mod}2)}{2}} \in Z(H).$$

Proof. Taking $h = 1$ in Proposition 2.6 yields

$$z_n = Z_n(1) = z_2 Z_{n-2}(1) = z_n z_{n-2}.$$

By induction $z_{2k} = z_2^k$. Since $z_1 = \sum \Lambda_1 S\Lambda_2 = 1$ it follows that $z_3 = z_2 z_1 = z_2$ and $z_{2k+1} = z_{2k} = z_2^k$. $\qquad\square$

In what follows we present z_2 in another form:

Theorem 2.8. *Let H be a semisimple Hopf algebra over an algebraically closed field k of characteristic 0. Then*

$$z_2 = \sum \Lambda_1^1 \Lambda_1^2 S\Lambda_2^1 S\Lambda_2^2 = \sum_i \frac{1}{d_i^2} E_i \in Z(H).$$

Proof. By Theorem 2.7, $z_2 \in Z(H)$. Hence,

$$z_2 = \sum (\Lambda_{ad}^1 \Lambda_1^2) S\Lambda_2^2 =$$

$$= \sum_i \frac{1}{d_i} \left\langle \chi_i, (\Lambda_{ad}^1 \Lambda_1^2) S\Lambda_2^2 \right\rangle E_i$$

$$= \sum_i \frac{1}{d_i} \left\langle \chi_i \leftharpoonup (\Lambda_{ad}^1 \Lambda_1^2), S\Lambda_2^2 \right\rangle E_i$$

$$= \sum_i \frac{1}{d_i^2} \left\langle \chi_i, \Lambda_{ad}^1 \Lambda_1^2 \right\rangle \left\langle \chi_i, S\Lambda_2^2 \right\rangle E_i \qquad \text{(by (5))}$$

$$= \sum_i \frac{1}{d_i^2} \left\langle \chi_i, \Lambda_1^2 \right\rangle \left\langle \chi_i, S\Lambda_2^2 \right\rangle E_i \qquad \text{(since } \chi_i \text{ is cocommutative)}$$

$$= \sum_i \frac{1}{d_i^2} \left\langle \chi_i s(\chi_i), \Lambda^2 \right\rangle E_i$$

$$= \sum_i \frac{1}{d_i^2} E_i \qquad \text{(by orthogonality of characters, (2))}$$

$\qquad\square$

As a consequence of Theorem 2.7 and Theorem 2.8 we obtain:

Corollary 2.9. Let H be a semisimple Hopf algebra. Then

$$z_n = \sum_i \frac{1}{d_i^{n-n(\mathrm{mod}2)}} E_i.$$

An important consequence is the following theorem:

Theorem 2.10. *Let H be a semisimple Hopf algebra over an algebraically closed field k of characteristic 0. Then H is commutative if and only if $z_n \in k$ for some $n \geq 2$, if and only if $z_n \in k$ for all $n \geq 2$.*

Proof. By Theorem 2.7 it is enough to consider z_{2n}. If $z_{2n} = \alpha \in k$, then $z_{2n} = \alpha \sum E_i$, hence by Corollary 2.9, $\alpha = \frac{1}{d_i^{2n}}$ for all i, implying all d_i are equal. Since $V_0 = k$ we have $d_0 = 1$ and so $d_i = 1$ for all i. This implies that H is commutative. The converse is trivial. $\qquad\square$

For a group G we have $z_2 = \sum_{g,h \in G} ghg^{-1}h^{-1}$ hence it is easy to see that any commutator belongs to $z_2 \leftharpoonup kG^*$. This gives rise to the following question:

Question 2.11. Let H be a semisimple Hopf algebra over an algebraically closed field k of characteristic 0. Is that true that

$$\mathrm{Com} = z_2 \leftharpoonup H^*?$$

While the answer to this question is unknown to us, we can show the following:

Theorem 2.12. *Let H be a semisimple Hopf algebra over an algebraically closed field k of characteristic 0 and let $n \geq 2$. Then the commutator sublagebra of H is the algebra generated by the left coideal*

$$H'_n = z_n \leftharpoonup H^*.$$

Proof. Since $z_2 \in H'$ by Proposition 2.2, and since $z_n = z_2^{\frac{n - n(\mathrm{mod}2)}{2}}$ by Theorem 2.7, it follows that $z_n \in H'$ and so $H'_n \subset H'$. Let N be the algebra generated by H'_n. Since H'_n is ad-stable by [CW4, Prop.2.5], it follows that N is a normal left coideal subalgebra of H, and $N \subset H'$.

Let $\pi : H \longrightarrow \overline{H}$ be the natural Hopf projection where $\overline{H} = H/HN^+$. Then since $\overline{\Lambda} = \pi(\Lambda)$ and $z_n = \sum \Lambda_1^1 \cdots \Lambda_1^n S\Lambda_2^1 \cdots S\Lambda_2^n \in N$, it follows that $\sum \overline{\Lambda_1^1 \cdots \Lambda_1^n S\Lambda_2^1 \cdots S\Lambda_2^n} = \overline{1}$. This implies by Theorem 2.10 that $\overline{H}$ is commutative, hence $H' \subset N$ and we are done. $\qquad\square$

The following summarizes properties of z_2 which are a direct consequence of this section.

Corollary 2.13. Let H be a semisimple hopf algebra and let $z_2 = \sum \Lambda_1^1 \Lambda_1^2 S\Lambda_2^1 S\Lambda_2^2$. Then:

1. $z_2 = \sum_i \frac{1}{d_i^2} E_i$ is an invertible central element of H and $Sz_2 = z_2$.

2. $z_2^k = z_2^l$, $k \neq l$, $k, l \in \mathbb{Z}$ if and only if H is commutative.

3. $\langle \chi_i, z_2 \rangle = \frac{1}{d_i}$ for all $0 \leq i \leq n - 1$.

4. $\{\Lambda, z_2^k\} \in Z(H)$ for all $k \in \mathbb{Z}$.

As for groups, we can define the iterated commutator $\{\{H, H\}, H\}$ for Hopf algebras H. In this case we can extend Proposition 2.2 as follows:

Proposition 2.14. *Let H be a semisimple Hopf algebra. Then the iterated commutator satisfies:*

$$\{\{H, H\}, H\} = k \Leftrightarrow \{H, H\} \subset Z(H).$$

Proof. Assume $\{H, H\} \subset Z(H)$. Since $\{H, H\}$ is a left coideal and $Z(H)$ is S-stable, we have for all $a, b, c \in H$,

$$\sum \{a, b\}_1 c_1 S\{a, b\}_2 Sc_2 = \sum \{a, b\}_1 S\{a, b\}_2 c_1 Sc_2 = \langle \varepsilon, abc \rangle.$$

Assume now $\{\{H, H\}, H\} = k$. Then for all $x \in \{H, H\}$, $h \in H$ we have:

$$SxSh = \sum Sh_1 Sx_1 \underbrace{x_2 h_2 Sx_3 Sh_3}_{\in \{\{H,H\},H\}} = ShSx.$$

Hence $x \in Z(H)$. $\qquad\square$

3. Analogues of counting functions for Hopf algebras

In what follows we indicate how counting functions can be realized from our point of view. Let G be a finite group. The since k is of characteristic 0, kG is a semisimple Hopf algebra with an idempotent integral $\Lambda = \frac{1}{|G|} \sum_{g \in G} g$. The projections $\{p_g \in kG^*\}$ defined by $\langle p_g, h \rangle = \delta_{g,h}$ form a basis for kG^* dual to the natural basis $\{g | g \in G\}$ of kG. Since the counting function N_w is a class function it follows that it is an element of $R(kG)$.

Observe that

$$N_w = \sum_{(g_1 ..., g_r) \in G^r} p_{w(g_1 ..., g_r)}.$$

The following formula appear in the literature (see e.g.[AV, (2)].) We show it here using a Hopf algebraic approach.

$$N_w =$$

$$= \frac{1}{d_i} \sum_i \langle N_w, E_i \rangle \chi_i \qquad \text{(by (4))}$$

$$= \sum_i \langle N_w, \Lambda \leftharpoonup s(\chi_i) \rangle \chi_i \qquad \text{(by (3))}$$

$$= \sum_i \langle s(\chi_i), \Lambda \leftharpoonup N_w \rangle \chi_i$$

$$= \sum_i \left\langle s(\chi_i), \frac{1}{|G|} \sum_{g \in G} g \leftharpoonup \sum_{(g_1 \dots, g_r) \in G^r} p_{w(g_1 \dots, g_r)} \right\rangle \chi_i$$

$$g = \sum_i \left\langle s(\chi_i), \frac{1}{|G|} \sum_{(g_1 \dots, g_r) \in G^r} w(g_1 \dots, g_r) \right\rangle \chi_i$$

This formula yields known formulations of some counting functions:

Example 3.1. 1. **Root functions**: if $w = x^m$ then N_w is the so called m-th root function counting the number of solutions in G to the equation $x^m = g$, $g \in G$. In this case

$$N_w = \sum_{g \in G} p_{g^m} = \frac{1}{G} \sum_i \left\langle s(\chi_i), \sum_{g \in G} g^m \right\rangle \chi_i = \frac{1}{|G|} \sum_i \left\langle \chi_i, \sum_{g \in G} g^m \right\rangle \chi_i.$$

The coefficient of χ_i is called the m-th Frobenius-Schur indicator.

2. **Frobenius function for commutators**: If $w = [x, y] = xyx^{-1}y^{-1}$ then

$$N_w = \sum_i \left\langle \chi_i, \frac{1}{|G|} \sum_{(g,h) \in G^2} ghg^{-1}h^{-1} \right\rangle \chi_i$$

But in terms of (17), $\sum_{(g,h) \in G^2} ghg^{-1}h^{-1} = |G|^2 z_2$, hence by Theorem 2.8 and Corollary 2.13.3,

$$N_w = \sum_i \langle \chi_i, |G| z_2 \rangle \chi_i = \sum_i \frac{|G|}{d_i} \chi_i$$

3. **generalized commutators** (see [Ta]): Similarly to the previous example, the counting function corresponding to the generalized commutator $w_n = [x_1, \dots, x_n] = x_1 \cdots x_n x_1^{-1} \cdots x_n^{-1}$ is given by:

$$N_{w_n} = \sum_i \frac{|G|^{n-1}}{d_i^{n-1-n(\text{mod}2)}} \chi_i.$$

Also $N_{w_{2k}}$ is the counting function corresponding to products of n commutators.

4. **Iterated commutators** (see [AV]): The counting function for the iterated commutator $w = [[x, y], z]$ can be calculated as follows:

$$N_w = \sum_i \left\langle s(\chi_i), \frac{1}{|G|} \sum_{(g_1, g_2, g_3) \in G^3} [[g_1, g_2], g_3] \right\rangle \chi_i = |G|^2 \sum_{i,j} \frac{1}{d_i d_j} \left\langle \chi_i s(\chi_i) \chi_j, \frac{1}{|G|} \sum_{g \in G} g \right\rangle \chi_i.$$

In Hopf algebras we do not know if we are counting anything. However, these functions have a meaning as elements in $R(H)$ related to commutators, bilinear forms, Casimir elements and higher Frobenius Schur indicators.

General root functions: A useful tool in studying the representations of Hopf algebras has been the introduction of Frobenius-Schur indicators, extending classical results of finite groups (see e.g [KaSZ]). For a Hopf algebra H define the m-th Sweedler power of $h \in H$ as $h^{[m]} = \sum h_1 h_2 \ldots h_m$, $m \geq 1$. For an irreducible character χ, the m-th Frobenius-Schur indicator is given by $\nu_m(\chi) = \langle \chi, \Lambda^{[m]} \rangle$.

If $H = kG$ then $\Lambda^{[m]} = \sum_{g \in G} g^m$ and so the function in Example 3.1.1 becomes

$$r_m = \sum_i \nu_m(\chi_i) \chi_i = \Psi(\Lambda^{[m]}).$$

Commutators and generalized commutators. Set

(19)
$$f_{rob} = \sum_i \frac{d}{d_i} \chi_i.$$

We show:

Proposition 3.2. *Let H be a semisimple Hopf algebra, let z_2 be the commutator as defined in (17) and let f_{rob} be defined as in (19). Then*

$$d\Psi(z_2) = f_{rob}.$$

Proof. Since $\Psi(E_i) = d_i s(\chi_i)$ by (3) and $z_2 = \sum_i \frac{1}{d_i^2} E_i$ by Theorem 2.8, it follows that

$$d\Psi(z_2) = f_{rob}.$$

$\square$

For groups, Frobenius counting function for product of n commutators can be extended to a formula counting $2n$-touples $(a_1, b_1, \ldots, a_n, b_n)$

such that $g = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1}$. ([KM]). This number is given explicitly by $f_n(g)$ where:

$$(20) \qquad\qquad f_n = \sum (\frac{d}{d_i})^{2n-1} \chi_i.$$

The formula above is related also to the convolution product defined on functions on G. Reminiscent of this products, a convolution product $\bullet$ inside H^* is defined as follows:

$$p \bullet p' = d\Psi(\Psi^{-1}(p)\Psi^{-1}(q)).$$

Variations of this product appear in [AN, CW1, PW]. This product should not be confused with the so called convolution product $*$ introduced by Kostant [Kos] which is the dual of the coproduct in H and is considered to be the usual product in H^*.

It turns out that for Hopf algebras the convolution product of f_{rob} is related to products of commutators as follows:

Theorem 3.3. *Let H be a semisimple Hopf algebra over an algebraically closed field k of characteristic 0. Then for any $l > 0$,*

$$d^{2l-1}\Psi(z_{2l}) = \sum_i \frac{d^{2l-1}}{d_i^{2l-1}} \chi_i = \underbrace{f_{rob} \bullet f_{rob} \cdots \bullet f_{rob}}_{l}.$$

Proof. For $l = 1$ this is shown in Proposition 3.2. In other words,

$$\Psi^{-1}(f_{rob}) = dz_2.$$

Since $z_2^2 = \sum \frac{1}{d_i^4} \chi_i$ by Theorem 2.8, it follows that $\Psi(z_2^2) = \sum \frac{1}{d_i^3} E_i$. Hence

$$f_{rob} \bullet f_{rob} = d\Psi(\Psi^{-1}(f_{rob})\Psi^{-1}(f_{rob})) = d^3\Psi(z_2^2) = \sum_i \frac{d^3}{d_i^3} \chi_i.$$

This proves the case of $l = 2$. The result follows now by induction. $\square$

Another point of view of the setup above is to realize the special commutators z_n as Casimir elements of certain symmetric bilinear on $Z(H)$ or H. The symmetric bilinear form on H defined in (9) induces a non-degenerate symmetric bilinear form on $Z(H)$:

$$\beta^Z(z, z') = \langle \lambda, zz' \rangle$$

Let $\{E_i\}$ be the set of central idempotents of H. Since $\langle \lambda, E_i E_j \rangle = \delta_{ij} d_i^2$, it follows that the Casimir element and the central Casimir element corresponding to β^Z are given by:

$$(21) \qquad \sum \frac{1}{d_i^2} E_i \otimes E_i \quad \text{and} \quad Cas_\beta^Z = \sum_i \frac{1}{d_i^2} E_i = z_2.$$

By using (3) it is straightforward to see that:

Lemma 3.4. *Let H be a semisimple Hopf algebra and let $t = \sum \alpha_i \chi_i$, $\alpha_i \in k$. Then:*

1. $t \leftharpoonup Z(H) = R(H)$ if and only if $\alpha_i \neq 0$ for all i.

2. If indeed $\alpha_i \neq 0$ for all i, then the central Casimir element Cas_t^Z of the corresponding bilinear form β_t is:

$$Cas_t^Z = \sum \frac{1}{\alpha_i d_i} E_i.$$

If moreover, $\alpha_i = \alpha_{i^*}$ (where α_{i^*} is the coefficient of $s(\chi_i)$), then:

$$\Psi(Cas_t^Z) = \sum_i \frac{1}{\alpha_i} \chi_i.$$

Summarizing results from Lemma 3.4, Theorem 3.3 and Corollary 2.9 we obtain the following realization of the counting functions in Example 3.1.3:

Corollary 3.5. *Let H be a semisimple Hopf algebra and let z_n, $n \geq 2$ be defined as in (17). Define the symmteric form β_n^Z on $Z(H)$ through:*

$$t_n = \sum_i d_i^{n-1-n(\mathrm{mod}2)} \chi_i.$$

Then the corresponding central Casimir element satisfies:

$$Cas_{\beta_n}^Z = z_n.$$

Moreover,

$$d^{n-1}\Psi(Cas_{\beta_n}^Z) = d^{n(\mathrm{mod}2)} \underbrace{f_{rob} \bullet \cdots \bullet f_{rob}}_{\frac{n-n(\mathrm{mod}2)}{2}} = \sum_i \frac{d^{n-1}}{d_i^{n-n(\mathrm{mod}2)}} \chi_i.$$

Note that $t_2 = \lambda$ and $Cas_{\beta_2}^Z = z_2$.

The elements z_n are related also to symmetric forms on H as follows:

Proposition 3.6. *Let H be a semisimple Hopf algebra and let z_n, $n \geq 2$ be defined as in (17). Then z_n is the central Casimir element of the symmetric form on H $\tilde{\beta}_n$ defind through:*

$$\tilde{t}_n = \sum_i d_i^{n+1-n(\mathrm{mod}2)} \chi_i, \qquad \tilde{\beta}_n^H(h, h') = \langle \tilde{t}_n, hh' \rangle$$

for all $h, h' \in H$.

The Higman map defined by this form (as in (8)) is

$$\tau_{\beta_n}(h) = Z_{n+1-n(\mathrm{mod}2)}(h).$$

Proof. Take $u_n = \sum_i d_i^{n-n(\mathrm{mod}2)} E_i = z_n^{-1}$. Then $\tilde{t}_n = \lambda \leftharpoonup u_n$. Since $u_n^{-1} = \sum \Lambda_1^1 \cdots \Lambda_1^n S\Lambda_1^2 \cdots S\Lambda_2^n$, it follows from (7) and (10) that the corresponding central Casimir element is $\Lambda_{ad} z_n = z_n$ since z_n is central.

The Higman map is given now by:

$$\tau_{\tilde{\beta}_n}(h) = \sum \Lambda_1 z_n h S\Lambda_2 = z_n(\Lambda \cdot h),$$

where the last equality follows since $z_n \in Z(H)$. By Theorem 2.7

$$z_n(\Lambda_{ad} h) = z_2^{\frac{n-n(\mathrm{mod}2)}{2}}(\Lambda \cdot h) = Z_{n+1-n(\mathrm{mod}2)}(h).$$

$\square$

Iterated commutators. Motivated by the counting function for iterated commutators for groups, (see [AV]), set:

$$(22) \qquad f_{\{\{x,y\},z\}} = d^2 \sum_{i,j} \frac{1}{d_i d_j} \langle \chi_i s(\chi_i)\chi_j, \Lambda \rangle \chi_i.$$

Then we have:

Theorem 3.7. *Let H be a semisimple hopf algebra over an algebraically closed fiels of characteristic* 0. *Then:*

$$f_{\{\{x,y\},z\}} = d^2 \sum_{i,j} \frac{1}{d_i} \langle \chi_i s(\chi_i), z_2 \rangle \chi_i = d^2 \Psi(\{z_2, \Lambda\}).$$

Proof. We have:

$$\langle \chi_i, \{z_2, \Lambda\}\rangle =$$
$$= \sum_{z_2, \Lambda} \langle \chi_i, z_{2_1} \Lambda_1 S z_{2_2} S\Lambda_2 \rangle$$
$$= \sum_{z_2, \Lambda} \langle \frac{1}{d_i}\chi_i, S z_{2_2}\rangle\langle \chi_i, z_{2_1}\rangle \qquad \text{(by (5) since } \Lambda \cdot S z_{2_2} \in Z(H))$$
$$= \frac{1}{d_i}\langle \chi_i s(\chi_i), z_2\rangle$$

Since $\{z_2, \Lambda\} \in Z(H)$ by Corollary 2.13.4, we have by (5),

$$\Psi(\{z_2, \Lambda\}) = \sum_i d_i \chi_i \leftharpoonup \{z_2, \Lambda\} = \sum_i \langle \chi_i, \{z_2, \Lambda\}\rangle\chi_i = \sum_i \frac{1}{d_i}\langle \chi_i s(\chi_i), z_2\rangle\chi_i.$$

This proves the second equality in the theorem. As for the first equality, since

$$z_2 = \sum_j \frac{1}{d_j^2} E_j = \Lambda \leftharpoonup \sum_j \frac{1}{d_j}\chi_j$$

it follows that:

$$\Psi(\{z_2, \Lambda\}) = d^2 \sum_{i,j} \frac{1}{d_i} \left\langle \chi_i s(\chi_i), \Lambda \leftharpoonup \frac{1}{d_j}\chi_j \right\rangle \chi_i = d^2 \sum_{i,j} \frac{1}{d_i d_j} \langle \chi_i s(\chi_i)\chi_j, \Lambda\rangle \chi_i.$$

By (22) we are done. $\qquad\square$

4. Commutators for almost cocommutative Hopf algebras

It is known ([Kac, Z]) that when H is semisimple then so is $R(H)$. Let $\{\frac{1}{d}\lambda = F_0, \dots F_{m-1}\}$ be a complete set of central primitive idempotents of $R(H)$, and let $\{f_0, \dots, f_{m-1}\}$ be primitive orthogonal idempotents in $R(H)$ so that $f_i F_j = \delta_{ij} f_i$. Define the **conjugacy class** $\mathfrak{C}_i$ as:

$$(23) \qquad \mathfrak{C}_i = \Lambda \leftharpoonup f_i H^*.$$

Then we have shown in [CW4] that:

$\mathfrak{C}_i$ *is an irreducible $D(H)$-module and moreover,*

$$H \cong \oplus_{i=0}^{m-1} \mathfrak{C}_i^{\oplus m_i}$$

as $D(H)$-modules.

We generalize also the notions of **Class sum** and of a representative of a conjugacy class as follows:

$$(24) \qquad C_i = \Lambda \leftharpoonup dF_i \qquad \eta_i = \frac{C_i}{\dim(f_i H^*)}.$$

We refer to η_i as a **normalized class sum**.

If $R(H)$ is commutative (which is equivalent to H being almost cocommutative [Ni]), then $\{F_i\}$ forms a basis of $R(H)$ and $f_i = F_i$ for all i. In this case $\{F_i\}, \{\eta_i\}$ is another pair of dual bases for $R(H)$ and $Z(H)$ respectively.

In this context z_2 has another realization.

Proposition 4.1. *Let H be a semisimple Hopf algebra and assume $R(H)$ is commutative. Then*

$$z_2 = \frac{1}{d} \sum_i \dim(F_i H^*)\eta_i S\eta_i$$

Proof. Since $\eta_i = \Psi^{-1}\left(\frac{d}{\dim(F_i H^*)}F_i\right)$, it follows that $\Psi(\eta_i) = \frac{d}{\dim(F_i H^*)}F_i$. A direct computation shows that $\langle F_i, \Lambda\rangle = \frac{\dim(F_i H^*)}{d}$ (see e.g. [CW3]). Hence

$$\langle \lambda, \eta_i S\eta_j\rangle = \langle \Psi(\eta_i), \eta_j\rangle = \left\langle \frac{d}{\dim(F_i H^*)}F_i, \Lambda \leftharpoonup \frac{d}{\dim(F_j H^*)}F_j \right\rangle = \delta_{ij}\frac{d}{\dim(F_i H^*)}.$$

Let β_2 be the bilinear form associated with $t_2 = \lambda$. Then $\{\eta_i\}$, $\{\frac{dim(F_i H^*)}{d} S\eta_i\}$ form a dual basis for β_2. The result follows now from Corollary 3.5. □

Observe that for a group G, we have g is a commutator if and only if $f_{rob}(g) \neq 0$. Moreover, $g \in G'$ if and only if $f_n(g) \neq 0$ for some $n \geq 1$, where f_n is as defined in (20). Similarly, g is an iterated commutator if and only if $f_{\{\{x,y\}z\}}(g) \neq 0$, where $f_{\{\{x,y\}z\}}$ is as defined in (22). Hence also for a class sum C, $f_{rob}(C) \neq 0$ (respectively $f_n(C) \neq 0$, $f_{\{\{x,y\}z\}}(C) \neq 0$) if and only if C is a sum of commutators (respectively $C \in kG'$, C is a sum of iterated commutators).

Since any ad-stable left coideal L is a $D(H)$-module and since the $\mathfrak{C}_i$ are all the irreducible $D(H)$-submodules of H, it follows that L is a sum of some of the $\mathfrak{C}_i$'s. As such L is generated as a coideal by some of the η_i's. Since H' is an ad-stable left coideal of H, the following question is relevant:

Question 4.2. Let H be a semisimple Hopf algebra over an algebraically closed field of characteristic 0, and let η be a normalized class sum. Is it true that $f_n(\eta) \neq 0$ if and only if $\eta \in H'$?

Recall H' is generated as an algebra by $\{z_n \leftharpoonup H^*\}$ (Theorem 2.12). Also recall that both H' and $\{z_n \leftharpoonup H^*\}$ are $D(H)$-modules hence are direct sums of conjugacy classes. As such they are obtained from certain normalized class sums. So a step in answering the question above is based on the following lemma:

Lemma 4.3. *Assume H is almost cocommutative and let $f = \alpha_i F_i \in R(H)$ and $P_f = \Psi^{-1}(f) \leftharpoonup H^*$. Then:*

$$\eta_j \in P_f \Longleftrightarrow \langle f, \eta_j \rangle \neq 0.$$

Proof. Assume $\eta_j \in P_f$. Then $\eta_j = \Psi^{-1}(f) \leftharpoonup q = \Psi^{-1}(fq)$. By the definition of η_j it follows that

$$fq = \frac{d}{\dim(F_i H^*)} F_j.$$

Since both η_j, $\Psi^{-1}(f) \in Z(H)$ it follows that $q \in R(H)$. Hence we have $q = \sum \beta_i F_i$, $\beta_i \in k$ and so $fq = \sum \alpha_i \beta_i F_i$. This implies that

$$\alpha_j \beta_j = \frac{d}{\dim(F_i H^*)},$$

in particular $\langle f, \eta_j \rangle = \alpha_j \neq 0$.

Conversely, assume $\langle f, \eta_j \rangle \neq 0$. Then $\alpha_j F_j = f F_j \neq 0$, which implies that $\eta_j \in P_f$. □

As a consequence we obtain:

Theorem 4.4. *Let H be an almost cocommutative semisimple Hopf algebra over an algebraically closed field of characteristic 0, let η_j be a normalized class and let f_n be as in (20). Then*

$$\eta_j \in z_2^n \leftharpoonup H^* \iff \langle f_n, \eta_j \rangle \neq 0.$$

Proof. The result follows from Lemma 4.3 with $f = f_n$. $\qquad\qquad\square$

Since $\Psi^{-1}(f_{\{\{x,y\},z\}}) = d^2\{z_2, \Lambda\}$ by Theorem 3.7, another consequence of Lemma 4.3 is:

Corollary 4.5. *If H is semisimple and almost cocommutative then:*

$$\eta_j \in \{z_2, \Lambda\} \leftharpoonup H^* \iff \langle f_{\{\{x,y\},z\}}, \eta_j \rangle \neq 0,$$

Remark 4.6. When $R(H)$ is commutative we can use the character table to write $\chi_i s(\chi_i)$ as a sum of the irreducible characters, and then compute $\langle \chi_i s(\chi_i), z_2 \rangle$ by using Corollary 2.13.3.

The known conjecture of Kaplansky is that if H is semisimple then $d_i | d$. It is known to be true for various families of Hopf algebras, in particular when H is a semisimple quasitriangular Hopf algebra [EG]. In this case we have:

Theorem 4.7. *Let H be a semisimple Hopf algebra over an algebraically closed field of characteristic 0, and assume $d_i | d$ for any irreducible H-module V_i. Then the functions f_{rob}, f_n and $f_{\{\{x,y\},z\}}$ given in (19), (20) and (22) are characters.*

Proof. The functions f_{rob} and f_n are clearly characters. We show it for $f_{\{\{x,y\},z\}}$ as well. Let $\chi_i s(\chi_i) = \sum m_{ki}\chi_k$, $m_{ki} \in \mathbb{Z}^+$. By Corollary 2.13.3, $\langle \chi_i s(\chi_i), z_2 \rangle = \sum_k \frac{m_{ki}}{d_k}$. Hence by Theorem 3.7,

$$f_{\{\{x,y\},z\}} = d^2 \sum_i \frac{1}{d_i} \langle \chi_i s(\chi_i), z_2 \rangle \chi_i = \sum_{i,k} \frac{d^2}{d_i d_k} m_{ki} \chi_i.$$

Since all the coefficients are integers we are done. $\qquad\qquad\square$

Remark 4.8. The root function for Hopf algebras is not necessarily a character for Hopf algebras satisfying the hypothesis of Theorem 4.7. In [IMaMo] the authors found groups G so that the Frobenius-Schur indicators of $D(G)$ are not integers and so the appropriate root function is not a character.

If (H, R) is also factorizable, more can be said. Recall [CW4] that when H is factorizable we have

$$(25) \qquad\qquad\qquad f_Q(\chi_i) = d_i \eta_i$$

for all $i > 0$, where $f_Q(p) = \sum \langle p, R^2 R^1 \rangle R^1 R^2$ is the Drinfeld map.

We can show:

Proposition 4.9. *Let H be a factorizable semisimple Hopf algebra. Then:*

1. Let $\chi_{ad} = \sum_i \chi_i s(\chi_i)$ be the character of the left adjoint representation of H. Then

$$f_Q(\chi_{ad}) = dz_2.$$

2. Let $F = f_Q \Psi$ and $F' = \Psi f_Q$ be the quantum Fourier Transforms defined on H and H^* respectively (see[LMj, CW1]). Then

$$F(z_2) = \sum_i \eta_i \qquad F'(\chi_{ad}) = f_{rob}.$$

Proof. 1. Follows from Proposition 4.1, (25), the fact that f_Q is multiplicative on $R(H)$ and $\dim(F_i H^*) = \dim(E_i H^*) = d_i^2$.

2. By Theorem 2.8, $z_2 = \sum_i \frac{1}{d_i^2} E_i$, hence we have:

$$f_Q \Psi(z_2) = f_Q \left(\sum_i \frac{1}{d_i} \chi_i \right) = \sum_i \eta_i.$$

The second part follows directly from part 1. $\qquad\qquad\square$

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